

ON WEIERSTRASS SEMIGROUP AT m POINTS ON CURVES OF THE TYPE $f(y) = g(x)$

A. S. CASTELLANOS AND G. TIZZOTTI

ABSTRACT. In this work we determine the so-called minimal generating set of the Weierstrass semigroup of certain m points on curves \mathcal{X} with plane model of the type $f(y) = g(x)$ over \mathbb{F}_q , where $f(T), g(T) \in \mathbb{F}_q[T]$. Our results were obtained using the concept of discrepancy, for given points P and Q on \mathcal{X} . This concept was introduced by Duursma and Park, in [4], and allows us to make a different and more general approach than that used to certain specific curves studied earlier.

1. INTRODUCTION

Let \mathcal{X} be a nonsingular, projective, geometrically irreducible curve of genus $g \geq 1$ defined over a finite field \mathbb{F}_q , let $\mathbb{F}_q[\mathcal{X}]$ be the field of rational functions and $Div(\mathcal{X})$ be the set of divisors on \mathcal{X} . For $f \in \mathbb{F}_q[\mathcal{X}]$, the divisor of f will be denoted by (f) and the divisor of poles of f by $(f)_\infty$. As follows, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where \mathbb{N} is the set of positive integers. Let P_1, \dots, P_m be distinct rational points on \mathcal{X} . The set

$$H(P_1, \dots, P_m) = \{(a_1, \dots, a_m) \in \mathbb{N}_0^m ; \exists f \in \mathbb{F}_q[\mathcal{X}] \text{ with } (f)_\infty = \sum_{i=1}^m a_i P_i\}$$

is called the *Weierstrass semigroup* at the points P_1, \dots, P_m . An element in $\mathbb{N}_0^m \setminus H(P_1, \dots, P_m)$ is called *gap* and the set $G(P_1, \dots, P_m) = \mathbb{N}_0^m \setminus H(P_1, \dots, P_m)$ is called *gap set* of P_1, \dots, P_m .

It is not difficult to see that the set $H(P_1, \dots, P_m)$ is a semigroup. The case $m = 1$ has been studied for decades with its relationship to coding theory, see e.g. [5], [6] and [17]. An important fact about this case is that the cardinality of $G(P_1)$ is g . The case $m = 2$ started to be studied by Kim, in [9], where several properties were presented. Others relevant papers in the case $m = 2$ are [2], [7], [8] and [14]. For $m > 2$, this semigroup has been determined for some specific curves as Hermitian and Norm-trace curves, see [12] and [13]. With increasing interest in this semigroup, many results have been produced with several applications, especially in coding theory. Examples of these applications can be found in [3], [5] and [11].

In this work we study the Weierstrass semigroup $H(P_1, \dots, P_m)$ for points on curves \mathcal{X} with plane model of the type $f(y) = g(x)$ over \mathbb{F}_q , where $f(T), g(T) \in \mathbb{F}_q[T]$. Our results were obtained using the concept of discrepancy, for given points P and Q on \mathcal{X} , see Definition 2.5. This concept was introduced by Duursma and Park in [4], and it was our main tool for obtain the set $\Gamma(P_1, \dots, P_m)$, called *minimal generating* of

$H(P_1, \dots, P_m)$, see Theorem 3.3. We observe that this approach is different from that used by Matthews in [12] and Matthews and Peachey in [13].

This paper is organized as follows. Section 2 contains general results about Weierstrass semigroup and discrepancy. In Section 3, we determine the minimal generating for the Weierstrass semigroup $H(P_1, \dots, P_m)$ for the curves \mathcal{X} with plane model of the type $f(y) = g(x)$ cited above. Finally, in Section 4 we present examples for certain specific curves.

2. WEIERSTRASS SEMIGROUP AND DISCREPANCY

Let \mathcal{X} be a non-singular, projective, irreducible, algebraic curve of genus $g \geq 1$ over a finite field \mathbb{F}_q .

Fix m distinct rational points P_1, \dots, P_m on \mathcal{X} . Define a partial order \preceq on \mathbb{N}_0^m by $(n_1, \dots, n_m) \preceq (p_1, \dots, p_m)$ if and only if $n_i \leq p_i$ for all i , $1 \leq i \leq m$.

For $\mathbf{u}_1, \dots, \mathbf{u}_t \in \mathbb{N}_0^m$, where, for all k , $\mathbf{u}_k = (u_{k1}, \dots, u_{km})$, we define the *least upper bound* (*lub*) of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_t$ in the following way:

$$\text{lub}\{\mathbf{u}_1, \dots, \mathbf{u}_t\} = (\max\{u_{11}, \dots, u_{t1}\}, \dots, \max\{u_{1m}, \dots, u_{tm}\}) \in \mathbb{N}_0^m.$$

For $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}_0^m$ and $i \in \{1, \dots, m\}$, we set

$$\nabla_i(\mathbf{n}) := \{(p_1, \dots, p_m) \in H(P_1, \dots, P_m) ; p_i = n_i\}.$$

Proposition 2.1. [12], Proposition 3] *Let $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}_0^m$. Then \mathbf{n} is minimal, with respect to \preceq , in $\nabla_i(\mathbf{n})$ for some i , $1 \leq i \leq m$, if and only if \mathbf{n} is minimal in $\nabla_i(\mathbf{n})$ for all i , $1 \leq i \leq m$.*

Proposition 2.2. [12], Proposition 6] *Suppose that $1 \leq t \leq m \leq q$ and $\mathbf{u}_1, \dots, \mathbf{u}_t \in H(P_1, \dots, P_m)$. Then $\text{lub}\{\mathbf{u}_1, \dots, \mathbf{u}_t\} \in H(P_1, \dots, P_m)$.*

Definition 2.3. Let $\Gamma(P_1) = H(P_1)$ and, for $m \geq 2$, define

$$\Gamma(P_1, \dots, P_m) := \{\mathbf{n} \in \mathbb{N}_0^m : \text{for some } i, 1 \leq i \leq m, \mathbf{n} \text{ is minimal in } \nabla_i(\mathbf{n})\}.$$

Lemma 2.4. [12], Lemma 4] *For $m \geq 2$, $\Gamma(P_1, \dots, P_m) \subseteq G(P_1) \times \dots \times G(P_m)$.*

In [12], Theorem 7, it is shown that, if $2 \leq m \leq q$, then $H(P_1, \dots, P_m) =$

$$\left\{ \begin{array}{l} \text{lub}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} \in \mathbb{N}_0^m : \quad \mathbf{u}_i \in \Gamma(P_1, \dots, P_m) \\ \quad \text{or } (u_{i_1}, \dots, u_{i_k}) \in \Gamma(P_{i_1}, \dots, P_{i_k}) \\ \quad \text{for some } \{i_1, \dots, i_k\} \subset \{1, \dots, m\} \text{ such that} \\ \quad i_1 < \dots < i_k \text{ and } u_{i_{k+1}} = \dots = u_{i_\ell} = 0, \\ \quad \text{where } \{i_{k+1}, \dots, i_m\} \subset \{1, \dots, \ell\} \setminus \{i_1, \dots, i_k\} \end{array} \right\}.$$

Therefore, the Weierstrass semigroup $H(P_1, \dots, P_m)$ is completely determined by $\Gamma(P_1, \dots, P_m)$. In [12], Matthews called the set $\Gamma(P_1, \dots, P_m)$ of *minimal generating* of $H(P_1, \dots, P_m)$.

In [4, Section 5], Duursma and Park introduced the concept of discrepancy as follows.

Definition 2.5. A divisor $A \in \text{Div}(\mathcal{X})$ is called a *discrepancy* for two rational points P and Q on \mathcal{X} if $\mathcal{L}(A) \neq \mathcal{L}(A - P) = \mathcal{L}(A - P - Q)$ and $\mathcal{L}(A) \neq \mathcal{L}(A - Q) = \mathcal{L}(A - P - Q)$.

The next result relates the concept of discrepancy with the set $\Gamma(P_1, \dots, P_m)$.

Lemma 2.6. Let $\mathbf{n} = (n_1, \dots, n_m) \in H(P_1, \dots, P_m)$. Then $\mathbf{n} \in \Gamma(P_1, \dots, P_m)$ if and only if the divisor $A = n_1P_1 + \dots + n_mP_m$ is a discrepancy with respect to P and Q for any two rational points $P, Q \in \{P_1, \dots, P_m\}$.

Proof. Let $\mathbf{n} = (n_1, \dots, n_m) \in \Gamma(P_1, \dots, P_m)$, then there is a rational function $f \in \mathbb{F}_q(\mathcal{X})$ such that $(f)_\infty = A = n_1P_1 + \dots + n_mP_m$. Let $P_i, P_j \in \{P_1, \dots, P_m\}$, with $i \neq j$. Suppose that $\mathcal{L}(A) = \mathcal{L}(A - P_i)$. Then we have that $f \in \mathcal{L}(A - P_i)$ and so $(f) + A - P_i \succeq 0$, contradiction because the pole divisor of f is A . Therefore $\mathcal{L}(A) \neq \mathcal{L}(A - P_i)$. Now, suppose that $\mathcal{L}(A - P_i) \neq \mathcal{L}(A - P_i - P_j)$. Then there is a rational function $g \in \mathcal{L}(A - P_i) \setminus \mathcal{L}(A - P_i - P_j)$. So $(g) + A - P_i \succeq 0$ and this implies that $(g)_\infty = s_1P_1 + \dots + s_{j-1}P_{j-1} + n_jP_j + s_{j+1}P_{j+1} + \dots + s_mP_m$, with $s_i \leq n_i - 1$ and $s_k \leq n_k$, for $k \in \{1, \dots, m\} \setminus \{i, j\}$. It follows that $(s_1, \dots, s_{j-1}, n_j, s_{j+1}, \dots, s_m) \in \nabla_j(\mathbf{n})$, this contradicts the fact that $\mathbf{n} \in \Gamma(P_1, \dots, P_m)$ is minimal in $\nabla_j(\mathbf{n})$. Similar considerations apply when initializing with P_j . Therefore, $A = n_1P_1 + \dots + n_mP_m$ is a discrepancy with respect to P and Q for any two rational points $P, Q \in \{P_1, \dots, P_m\}$. Conversely, suppose that $\mathbf{n} = (n_1, \dots, n_m) \in H(P_1, \dots, P_m)$ and that the divisor $A = n_1P_1 + \dots + n_mP_m$ is a discrepancy with respect to any $P_i, P_j \in \{P_1, \dots, P_m\}$, with $i \neq j$. Since $\mathbf{n} = (n_1, \dots, n_m) \in H(P_1, \dots, P_m)$, there is a rational function $f \in \mathbb{F}_q(\mathcal{X})$ such that $(f)_\infty = n_1P_1 + \dots + n_mP_m$. Suppose that $\mathbf{n} = (n_1, \dots, n_m)$ is not minimal in $\nabla_i(\mathbf{n})$ for some $i \in \{1, \dots, m\}$. Then, there exist a rational function $f_i \in \mathbb{F}_q(\mathcal{X})$ such that $(f)_\infty = s_1P_1 + \dots + s_mP_m$ with $\mathbf{s} = (s_1, \dots, s_m) \in \nabla_i(\mathbf{n})$, $\mathbf{s} \neq \mathbf{n}$ and $\mathbf{s} \preceq \mathbf{n}$. Therefore, $s_i = n_i$ and $s_j < n_j$ for some $j \neq i$. It follows that $f_i \in \mathcal{L}(A - P_j)$ and $f_i \notin \mathcal{L}(A - P_i)$, this contradicts the fact that $\mathcal{L}(A - P_j) = \mathcal{L}(A - P_i)$. \square

3. WEIERSTRASS SEMIGROUP $H(P_1, \dots, P_m)$ FOR CERTAIN TYPES OF CURVES

Consider a curve \mathcal{X} over \mathbb{F}_q given by affine equation $f(y) = g(x)$, where $f(T), g(T) \in \mathbb{F}_q[T]$, $\deg(f(y)) = a$ and $\deg(g(x)) = b$, with $\gcd(a, b) = 1$. Suppose that \mathcal{X} is absolutely irreducible with genus $g = (a - 1)(b - 1)/2$.

Let P_1, P_2, \dots, P_{a+1} be $a + 1$ distinct rational points such that

$$(3.1) \quad aP_1 \sim P_2 + \dots + P_{a+1},$$

and

$$(3.2) \quad bP_i \sim bP_j, \text{ for all } i, j \in \{1, 2, \dots, a+1\},$$

where “ \sim ” represent the equivalence of divisors. Note that $H(P_1) = \langle a, b \rangle$.

Examples of such curves can be found in [1], [10], [12] and [13].

Let $1 \leq m \leq a+1 \leq q$. For

$$(3.3) \quad t + \sum_{j=2}^m s_j = a+1 - m,$$

we have that the equivalences (3.1) and (3.2) yield

$$(3.4) \quad (tb - ia)P_1 + \sum_{j=2}^m (s_j b + i)P_j \sim \sum_{j=m+1}^{a+1} (b - i)P_j.$$

Observe that all coefficients are positive for

$$(3.5) \quad 0 < ia < tb, \quad s_j \geq 0.$$

Note that $0 < t \leq a$ and thus $0 < i < b$.

From the basic equivalence (3.1) and (3.2) we have that

$$(3.6) \quad A = (b - i)(aP_1 - P_2 - \dots - P_m) \sim \sum_{j=m+1}^{a+1} (b - i)P_j.$$

By (3.2) the divisor on the left can be replaced with an efficient equivalent divisor of the follow form

$$(3.7) \quad A' = ((a+1-m)b - ia)P_1 + i(P_2 + \dots + P_m) \sim \sum_{j=m+1}^{a+1} (b - i)P_j.$$

Thus, by redistributing $a+1-m = s+t$ over P_1 and P_2 , the divisors in (3.4), with $s = \sum_{j=2}^m s_j$, contain the special representative

$$(3.8) \quad (tb - ia)P_1 + (sb + i)P_2 + i(P_3 + \dots + P_m) \sim \sum_{j=m+1}^{a+1} (b - i)P_j.$$

The other divisors with same t, s and i are easy obtained from (3.8) by distributing s in all possible ways over P_2, \dots, P_{m+1} such that $s = \sum_{j=2}^m s_j$.

Proposition 3.1. *Let b and i be as above. Then, the divisor $(b-i)(aP_1 - P_2 - \dots - P_m)$ is a discrepancy with respect to P and Q for any two distinct points $P, Q \in \{P_1, \dots, P_m\}$.*

Proof. It suffices to prove the claim for the equivalent but effective divisor $A' = ((a + 1 - m)b - ia)P_1 + i(P_2 + \cdots + P_m)$. The divisors A and A' appear on the left side of the divisor equivalences (3.6) and (3.7), respectively. The equivalence of effective divisors in (3.7) gives a rational function $f \in L(A')$ with pole divisor equal to A' . Thus $L(A') \neq L(A' - P)$ for any point P . To prove that $L(A' - P) = L(A' - P - Q)$ we consider the equivalent statement $L(K + P + Q - A') \neq L(K + P - A')$. For the choice of canonical divisor $K = (ab - a - b - 1)P_1$,

$$\begin{aligned} K + P + Q - A' &= (ab - a - b - 1)P_1 + P + Q - (a + 1 - m)b - ia)P_1 \\ &\quad - i(P_2 + \cdots + P_m) \\ &= (i - 1)a + (m - 2)b - 1)P_1 + P + Q - i(P_2 + \cdots + P_m). \end{aligned}$$

Consider first the case $P_1 \in \{P, Q\}$. Without loss of generality we may assume that $P = P_1, Q = P_2$. Let f_2, \dots, f_m be the functions with divisors

$$(f_j) = \begin{cases} -aP_1 + (P_2 + \cdots + P_{a+1}), & j = 2 \\ b(P_j - P_1), & j > 2 \end{cases}$$

Then $f_2^{i-1}f_3 \cdots f_m \in L(K + P + Q - A') \setminus L(K + Q - A')$. Thus $L(A' - Q) = L(A' - P - Q)$. From $L(A') \neq L(A' - Q) = L(A' - P - Q)$ and $L(A') \neq L(A' - P)$ it follows that also $L(A' - P) = L(A' - P - Q)$. Consider next the case $P_1 \notin \{P, Q\}$, say $P = P_2, Q = P_3$. Thus, we have that $f_2^{i-1}f_4 \cdots f_m \in L(K + P + Q - A') \setminus L(K + Q - A')$. As before, it follows that $L(A' - Q) = L(A' - P - Q)$ and that $L(A' - P) = L(A' - P - Q)$. \square

Corollary 3.2. *Let $a, b, t, i, s_2, \dots, s_m$ be as above. Then, the divisor $(tb - ia)P_1 + \sum_{j=2}^m (s_j b + i)P_j$ is a discrepancy with respect to P and Q for any two distinct points $P, Q \in \{P_1, \dots, P_m\}$.*

Proof. Follows directly from the previous proposition and equations (3.4) and (3.8). \square

Theorem 3.3. *Let \mathcal{X} and P_1, P_2, \dots, P_{a+1} be as above. For $2 \leq m \leq a + 1$, let*

$$S_m = \left\{ (tb - ia, s_2 b + i, \dots, s_m b + i); t + \sum_{j=2}^m s_j = a + 1 - m, 0 < ia < tb, s_j \geq 0 \right\}.$$

Then, $\Gamma(P_1, \dots, P_m) = S_m$.

Proof. By Corollary 3.2, the divisor $(tb - ia)P_1 + \sum_{j=2}^m (s_j b + i)P_j$ is a discrepancy with respect to P and Q for any two distinct points $P, Q \in \{P_1, \dots, P_m\}$. So, by Lemma 2.6, follows that $S_m \subseteq \Gamma(P_1, \dots, P_m)$.

Next, we show that $\Gamma(P_1, \dots, P_m) \subseteq S_m$. Let $\mathbf{n} = (n_1, \dots, n_m) \in \Gamma(P_1, \dots, P_m)$. By Lemma 2.4, $\mathbf{n} = (n_1, \dots, n_m) \in G(P_1) \times G(P_2) \times \cdots \times G(P_m)$.

As $H(P_1) = \langle a, b \rangle$, from Lemma 1 in [16] we have that $n_1 = ab - i_1 a - j_1 b = (a - j_1)b - i_1 a$, for some $i_1, j_1 \in \mathbb{N}$. Let $\lambda = a - j_1$. Note that $0 < j_1 < a$ and $0 < ai_1 < b\lambda$.

By Equation (3.2), follows that $b \in H(P_\ell)$, for $2 \leq \ell \leq m$. So, we have that $n_\ell = s_\ell b + i_\ell$, where $0 < i_\ell < b$ and $s_\ell \geq 0$.

Let $i = \min\{i_\ell : 2 \leq \ell \leq m\}$. By Equation (3.1), for each $\ell = 2, \dots, m$, there is a rational function h_ℓ such that $(h_\ell) = bP_1 - bP_\ell$. By Equation (3.1), there is a rational function g such that $(g) = P_2 + \dots + P_{a+1} - aP_1$.

Let $\omega = g^{b-i} \cdot \prod_{\ell=2}^m h_\ell^{s_\ell+1}$. Then, $(\omega)_\infty = ((a - \sum_{\ell=2}^m (s_\ell + 1))b - ia)P_1 + (s_2b + i)P_2 + \dots + (s_mb + i)P_m$. Taking $t = a - \sum_{\ell=2}^m (s_\ell + 1)$, by Corollary 3.2, $(\omega)_\infty$ is a discrepancy with respect to P and Q for any two distinct points $P, Q \in \{P_1, \dots, P_m\}$. So, by Lemma 2.6, $\mathbf{w} = (tb - ia, s_2b + i, \dots, s_mb + i) \in \Gamma(P_1, \dots, P_m)$. Now, we know that $i = i_k$, for some $k \in \{2, \dots, m\}$. Then, $\mathbf{w} \in \nabla_k(\mathbf{n})$ and by minimality of \mathbf{w} and \mathbf{n} follows that $\mathbf{w} = \mathbf{n}$ and $\Gamma(P_1, \dots, P_m) \subseteq S_m$. □

4. EXAMPLES

Example 4.1. Let $\mathcal{X}_{q^{2r}}$ be the curve defined over $\mathbb{F}_{q^{2r}}$ by the affine equation

$$y^q + y = x^{q^r+1},$$

where r is an odd positive integer and q is a prime power. Note that \mathcal{X}_{q^2} is just the Hermitian curve. The curve $\mathcal{X}_{q^{2r}}$ has genus $g = q^r(q-1)/2$, a single point at infinity, namely $P_1 = (0 : 1 : 0)$, and others q^{2r+1} rational points. It is important to observe that $\mathcal{X}_{q^{2r}}$ is a quotient of the Hermitian curve and thus is a maximal curve over $\mathbb{F}_{q^{2r}}$. The Weierstrass semigroup $H(P_1, P_2)$ was studied in [15] and more details about this curve can be found in [10].

Let y_1, \dots, y_q be the solutions in $\mathbb{F}_{q^{2r}}$ to $y^q + y = 0$. Let $P_2 = (0, y_1), P_3 = (0, y_2), \dots, P_{q+1} = (0, y_q)$. Since $(x) = P_2 + \dots + P_{q+1} - qP_1$ and $(y - y_j) = (q^r + 1)P_{j+1} - (q^r + 1)P_1$, for all $j = 1, \dots, q$, we have that

$$(4.1) \quad qP_1 \sim P_2 + \dots + P_{q+1},$$

and

$$(4.2) \quad (q^r + 1)P_i \sim (q^r + 1)P_j, \text{ for all } i, j \in \{1, 2, \dots, q+1\},$$

For this curve, take $m = 4$, $q = 5$ and $r = 3$ we have that $a = 5$ and $b = 126$. Then by Theorem (3.3) follows that $\Gamma(P_1, P_2, P_3, P_4)$ consists of the following 125 elements

$$\begin{aligned} (126, 0, 0, 126) &+ i(-5, 1, 1, 1), \quad i = 1, \dots, 25, \\ (126, 0, 126, 0) &+ i(-5, 1, 1, 1), \quad i = 1, \dots, 25, \\ (126, 126, 0, 0) &+ i(-5, 1, 1, 1), \quad i = 1, \dots, 25, \\ (252, 0, 0, 0) &+ i(-5, 1, 1, 1), \quad i = 1, \dots, 50. \end{aligned}$$

Example 4.2. Let a Kummer extensions given by $y^b = g(x) = \prod_{i=1}^a (x - \alpha_i)$ where $g(x)$ is a separable polynomial over \mathbb{F}_q of degree a and $\gcd(a, b) = 1$. We know that these

curves has a single point P_1 at infinity and has genus $(b-1)(a-1)/2$, see [1]. Then we have the following divisors in $F(\mathcal{X})$:

- (1) $(x - \alpha_i) = bP_i - bP_1$ for every i , $2 \leq i \leq a+1$,
- (2) $(y) = P_2 + \cdots + P_{a+1} - aP_1$,

For $a = 5$ and $b = 7$ we have that

$$\begin{aligned}\Gamma(P_1, P_2) &= \{(23, 1), (18, 2), (13, 3), (8, 4), (3, 5), (16, 8), \\ &\quad (11, 9), (6, 10), (1, 11), (9, 15), (4, 16), (2, 22)\} . \\ \Gamma(P_1, P_2, P_3) &= \{(2, 8, 8), (2, 15, 1), (2, 0, 15), (9, 8, 1), (9, 1, 8), \\ &\quad (4, 9, 2), (4, 2, 9), (16, 1, 1), (11, 2, 2), (6, 3, 3), (1, 4, 4)\} . \\ \Gamma(P_1, P_2, P_3, P_4) &= \{(2, 8, 1, 1), (2, 1, 8, 1), (2, 1, 1, 8), (9, 1, 1, 1), (4, 2, 2, 2)\} . \\ \Gamma(P_1, P_2, P_3, P_4, P_5) &= \{(2, 1, 1, 1, 1)\} .\end{aligned}$$

5. ACKNOWLEDGMENT

The authors would like to thank I. Duursma for very useful suggestions and comments that improved the results and the presentation of this work.

REFERENCES

- [1] M. Abdón, H. Borges and L. Quoos, *Weierstrass points on Kummer extensions*, (Nov. 2015) [Online]. Available: <http://arxiv.org/abs/1308.2203>.
- [2] E. Ballico, *Weierstrass points and Weierstrass pairs on algebraic curves*, Int. J. Pure Appl. Math., 2 (2002), 427-440.
- [3] C. Carvalho e F. Torres, *On Goppa codes and Weierstrass gaps at several points*, Des. Codes Cryptogr., 35(2) (2005), 211-225.
- [4] I. Duursma and S. Park, *Delta sets for divisors supported in two points*, Finite Fields and Their Applications, 18 (5), 2012, 865-885.
- [5] A. Garcia, S. J. Kim, and R. F. Lax, *Consecutive Weierstrass gaps and minimum distance of Goppa codes*, J. Pure Appl. Algebra, 84 (1993), 199-207.
- [6] T. Høholdt, J. van Lint, R. Pellikaan, *Algebraic geometry codes*, V.S. Pless, W.C. Huffman (Eds.), Handbook of Coding Theory, v. 1, Elsevier, Amsterdam, 1998.
- [7] M. Homma e S. J. Kim, *Goppa codes with Weierstrass pairs*, J. Pure Appl. Algebra, 162, 2001, 273-290.
- [8] M. Homma, *The Weierstrass semigroup of a pair of points on a curve*, Arch. Math., 67, 1996, 337-348.
- [9] S.J. Kim, *On index of the Weierstrass semigroup of a pair of points on a curve*, Arch. Math., 62, 1994, 73-82.
- [10] S. Kondo, T. Katagiri and T. Ogihara *Automorphism groups of one-point codes from the curves $y^q + y = x^{q^r+1}$* , IEEE Trans Inform Theory, 47, 2001, 2573 - 2579.
- [11] G. L. Matthews, *Weierstrass pairs and minimum distance of Goppa codes*, Designs, Codes and Cryptography, 22, 2001, 107-121.

- [12] G. L. Matthews, *The Weierstrass semigroup of an m -tuple of collinear points on a Hermitian curve*, Lecture note in Comput. Sci., Springer, Berlin, 2948, 2004, 12 - 24.
- [13] G. L. Matthews and J. D. Peachey, *Minimal generating sets of Weierstrass semigroups of certain m -tuples on the norm-trace function field*, Contemporary Mathematics, 1 518, 2010, 315-326.
- [14] C. Munuera, G. Tizziotti and F. Torres, *Two-Points Codes on Norm-Trace Curves*, Second International Castle Meeting, ICMCTA 2008 (A. Barbero Ed.), Lecture Notes in Comput. Sci., Springer-Verlag Berlin Heidelberg, 5228, 2008, 128-136.
- [15] A. Sepúlveda, G. Tizziotti, *Weierstrass semigroup and codes over the curve $y^q + y = x^{q^r+1}$* , Advances in Mathematics of Communications, 8, 2014, 67-72.
- [16] J. C. Rosales, *Fundamental gaps of numerical semigroups generated by two elements*, Linear Algebra and its Applications, 405, 2005, 200-208
- [17] H. Stichtenoth, *Algebraic Function Fields and Codes*, Berlin, Germany: Springer, 1993.